MAC 2312 – Calculus II
Guided Notes

Section 11.8	Power Series

This chapter covers
1) power series representation
2) interval of convergence
3) radius of convergence

form of a power series

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

centered at a or about a

our goals
1) create the series
2) determine for what values of x the series will converge

process Often we use ratio or root test to find the initial interval. Then we must test each endpoint using the tests we explored in 11.1 – 11.7.

Example 1

Find all the values of x for which the power series is absolutely convergent:

$$1 + \frac{1}{5}x + \frac{2}{5^2}x^2 + \frac{3}{5^3}x^3 + \cdots + \frac{n}{5^n}x^n + \cdots$$

$$\boxed{1 + \sum_{n=1}^{\infty} \frac{n}{5^n} x^n}$$

use ratio test to test for absolute convergence

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \quad \text{with} \quad a_n = \frac{n}{5^n} x^n$$

$$\lim_{n\to\infty} \left| \left(\frac{(n+1)}{5^{n+1}} \cdot x^{n+1} \right) \cdot \frac{5^n}{n x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n+1}{n} \cdot \frac{x^{n+1}}{x^n} \cdot \frac{5^n}{5^{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n+1}{n} \cdot \frac{x}{5} \right| = \left| \frac{x}{5} \right| = \frac{1}{5}|x|$$

By ratio test, the series is absolutely convergent if

$$\frac{1}{5}|x| < 1 \Rightarrow |x| < 5 \Rightarrow -5 < x < 5$$

The problem $\le$?? $\le$??

so we test each endpoint separately.

$\boxed{x = +5}$ $\quad a_n = \frac{n}{5^n} \cdot (5)^n = n$

so we are evaluating $\sum_{n=1}^{\infty} n = 1 + 2 + 3 + \cdots$

$1 + \sum_{n=1}^{\infty} n = 1 + 1 + 2 + 3 + \cdots + n + \cdots$

divergent since $\lim_{n \to \infty} \sum_{n=1}^{\infty} n \neq 0$

$\boxed{x = -5}$ $\quad a_n = \frac{n}{5^n}(-5)^n = n(-1)^n$

so we are evaluating $\sum_{n=1}^{\infty} (-1)^n \cdot n$

$\underbrace{1 + \sum_{n=1}^{\infty} (-1)^n \cdot n}$ $= 1 - 1 + 2 - 3 + \cdots + n(-1)^n + \cdots$

By alternating series test $= \sum_{n=1}^{\infty} (-1)^n \cdot n \neq 0$

so also divergent

Radius of Convergence

$|x| < 5$

$\Rightarrow \boxed{5}$

Final conclusion

Interval of convergence

$\boxed{(-5, 5)}$

Example 2

Find all values of x for which

$$1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + \cdots$$

is absolutely convergent.

$n=0$ $\quad \frac{1}{0!}x^0 = 1$, so no need to adjust the limits of the summation

$$a_n = \frac{1}{n!}x^n$$

Use Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1}{(n+1)!}(x)^{n+1} \cdot \frac{n!}{x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n} \cdot \frac{\cancel{n!}}{(n+1)\cancel{n!}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = \lim_{n\to\infty}\frac{1}{n+1}|x| = \boxed{0} < 1$$

so by ratio test, this power series is convergent for all values of x.

r = radius of convergence

Some summary theorems

① If a power series $\sum a_n x^n$ converges for a non-zero number c, then it is absolutely convergent whenever $|x| < |c|$.

② If a power series $\sum a_n x^n$ diverges for a non-zero number d, then it diverges when $|x| > |d|$

If $\sum a_n x^n$ is a power series, then one of the following is true:

1) The series converges only when $x = 0$
2) The series converges for all values of x
3) There is a positive number r such that the series is absolutely convergent if $|x| < |r|$ and divergent if $|x| > |r|$

note the values of $\pm r$ must be tested individually

possibilities

$(-r, r)$ $[-r, r)$ $(-r, r]$ $[-r, r]$

Adjustments for $\sum a_n(x-c)^n$

3 possibilities

1. The series converges only if $x-c=0$
   or $x=c$

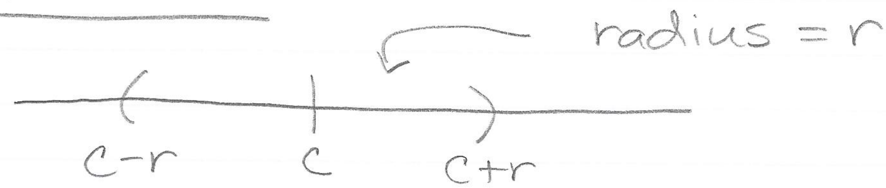
2. The series is absolutely convergent for all x

3. There is a number r such that the series is absolutely convergent if

   $|x-c|<r$ and divergent if $|x-c|>r$

again test each endpoint individually.

possibilities



$(c-r, c+r)$ $[c-r, c+r)$ $(c-r, c+r]$ $[c-r, c+r]$

Example 3

Find the interval of convergence for

$$\sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{n+1}\right)(x-3)^n$$

Ratio test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(n+1)+1} \cdot \frac{(x-3)^{n+1}}{1} \cdot \frac{(n+1)}{(-1)^n} \cdot \frac{1}{(x-3)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n+1}{n+2} \quad \frac{(-1)^{n+1}}{(-1)^n} \cdot \frac{(x-3)(x-3)^n}{(x-3)^n} \right|$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n+2}\right) \left| (-1)(x-3) \right| \quad = |x-3|$$

becomes 1

$$|x-3| < 1$$
$$-1 < x-3 < 1$$
$$-1+3 < x < 1+3$$
$$\boxed{2 < x < 4}$$

base interval of convergence is $(2, 4)$

The power series is absolutely convergent for these values.

$\boxed{\text{diverges for } x > 4, \; x < 2}$

Now test the end points.
2 & 4

Final Results: $\boxed{(2,4)}$

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$\boxed{x=4}$ $a_n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}(2-3)^n = \sum_{n=0}^{\infty} \frac{(-1)^n(+1)^n}{(n+1)}$

$= \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1}}$

use alternating series test

$a_k \geq a_{k+1} > 0$ for every k the pattern $\frac{1}{2} > \frac{1}{3}$ etc.

and $\lim_{n\to\infty} \left(\frac{1}{n+1}\right) = 0$

so $\boxed{\text{at } x=4}$, the series is $\boxed{\text{convergent.}}$

$\boxed{x=2}$ $a_n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}(2-3)^n = \sum_{n=0}^{\infty} \frac{(-1)^n(-1)^n}{n+1}$

$= \sum_{n=0}^{\infty} \frac{(-1)^{2n}}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1}$

use comparison test compare to $\sum \frac{1}{n}$
or **limit comparison test** harmonic

$b_n = \frac{1}{n}$ $a_n = \frac{1}{n+1}$ $\boxed{\text{divergent}}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{n+1} \cdot \frac{n}{1} = 1 > 0$

so since $\sum \frac{1}{n}$ diverges, so does $\sum \frac{1}{n+1}$

11.8 Power Series

Here are some for you to try:

Find the interval of convergence for each power series:

	answer
① $\sum_{n=0}^{\infty} \frac{n^2}{2^n} x^n$	$(-2, 2)$
② $\sum_{n=0}^{\infty} \frac{n+1}{10^n} (x-4)^n$	$(-6, 14)$
③ $\sum_{n=0}^{\infty} \frac{1}{(-4)^n} x^{2n+1}$	$(-2, 2)$
④ $\sum_{n=1}^{\infty} \frac{\ln n}{e^n} (x-e)^n$	$(0, 2e)$
⑤ $\sum_{n=0}^{\infty} \frac{3^{2n}}{n+1} (x-2)^n$	$\left[\frac{17}{9}, \frac{19}{9}\right)$
⑥ $\sum_{n=2}^{\infty} \frac{n}{n^2+1} x^n$	$[-1, 1)$